International Journal of HEAT AND FLUID FLOW

International Journal of Heat and Fluid Flow 20 (1999) 166–171

Friction dominated, subsonic compressible flow in channels of slowly varying cross section

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Received 30 January 1998; accepted 2 October 1998

Abstract

Friction dominated, subsonic compressible flow in micro-channels of slowly varying cross-section is treated by developing a perturbation scheme which yields equations of "lubrication" approximation as the first order one. In this context several characteristic problems are encountered, such as isothermal flow, non-isothermal flow with prescribed wall temperature, and non-isothermal flow with prescribed wall heat flux, which includes the adiabatic wall problem as a special case. In all these problems pressure drops to zero while velocity components increase indefinitely at a finite distance along the channel – the phenomenon called "mathematical" choking observed for the first time in isothermal flow between parallel side walls, Schwartz, L.W., 1987. A perturbation solution for compressible viscous channel flows. J. Engrg. Math. 21, 69–86. The problem with prescribed wall heat flux is characterised, as by Shajii, A., Freidberg, J.P., 1996. Theory of law Mach number compressible flow in a channel. J. Fluid. Mech. 313, 131–145, by the existence of a critical heat flux above which the steady flow cannot be maintained. For the same mass flow rate this flux is considerably greater in divergent channels, than in convergent ones. These and other results obtained show how the prevailing viscosity may dramatically alter the flow characteristics in the problem considered, in comparison with more conventional high Reynolds number flows.

1. Introduction

Compressible gas flows at moderately high values of the Reynolds number in channels of constant or variable cross-section are frequently met in several branches of contemporary techniques and have attracted the attention of several researchers in the literature recently. Since gases usually flow with relatively high velocities, moderately high values of the Reynolds number can be maintained in channels of extremely small width only, in so-called micro-channels. Shear-driven flows in micro-channels occur in externally pressurised thrust bearings and micromotors, while pressure-driven flows find very useful applications in problems of integrated cooling of electronic circuits and superconducting magnets, in cryo-coolers for infra-red detectors and diode lasers, in high-frequency fluidic control elements, etc. Thus, both shear-driven and pressure-driven micro-channel flows represent an important constituent part of what is now called Micro-Electro–Mechanical-Systems (MEMS) technology (see two excellent review papers by Ho and Tai, 1996, and Beskok et al., 1996).

The main feature of all these flows is that the effect of viscosity is spread over the whole cross-section of the channel, so that it either competes the inertia, as in the classical boundary layer theory, or prevails over it, as in the hydrodynamic lubrication theory. It is shown by Crnojevic and Djordjevic (1997) that viscosity competes inertia for high subsonic or supersonic flow in a micro-channel. In such a case the problem is described by classical boundary layer equations, but it is defined inversely as the width of the channel, that plays the role of the boundary layer thickness, is given, while the pressure and other flow quantities, including the centreline velocity, that plays the role of the free stream velocity, are required. In spite of this "inconvenience", however, it is shown in the paper that certain modified Stewartson's transformations can be constructed in order to convert the governing equations into a fully incompressible form. This modification is caused by the fact that the flow along the centreline of the channel is not isentropic. The results obtained show dramatic effects that viscosity may have upon the flow characteristics in this case, in comparison with more conventional high Reynolds number flows. For low subsonic flow in a micro-channel viscosity prevails over inertia, so that the first order equations governing such a problem represent actually a "lubrication" approximation. An isothermal flow between parallel plates of this type was treated by Schwartz (1987), while the problem with applied heat flux over a straight circular channel was treated by Shajii and Freidberg (1996) in conjunction with the possible cooling of large-scale superconducting magnets that might be used in future magnetic fusion experiments. In these papers, several interesting and unexpected flow phenomena were discovered, such as "mathematical" choking in which pressure drops to zero over a finite length of the channel, the existence of some critical heat flux/mass flow rate above/below which steady flow cannot be maintained, etc.

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This paper represents a relatively simple extension of the papers by Schwartz (1987) and Shajii and Freidberg (1996) to account for the effect of variable width of the channel on the friction dominated, low Mach number subsonic flow. In order to treat the problem analytically, we introduce the following two small parameters of the same order of magnitude: maximum angle of the channel wall to the channel axis and (Mach number)^2/Reynolds number, and develop a regular perturbation scheme which yields a set of non-linear equations at each order. Equations of at least first two orders can be solved analytically. We treat separately isothermal flow and several non-isothermal cases of flow. In the isothermal flow case we discuss the effect of variable width of the channel on the "mathematical" choking, and study this phenomenon in more details by analysing the solutions of the second order equations. We find that singularities appearing in these solutions at the critical cross-section of the channel are stronger than in the first order solutions, and point out a possible way for the resolution of the question concerned with the finite values of physical quantities in this cross-section.

In the non-isothermal flow case we focus our attention to the influence of some applied heat flux on the flow characteristics, and in particular on the mass flow rate through the channel. In the realistic case in which the dependence of the transport coefficients on the temperature is pronounced we find, as in Shajii and Freidberg (1996), the existence of two bifurcated solutions, one of which is physically not acceptable. Also, steady solutions for the given pressure drop exist only if the heat flux lies below a critical value, and the mass flow rate is above a corresponding critical value. Both critical values are evaluated and their dependence on the dimensions of the exit cross-section of the channel discussed. The results show a very sharp increase of both critical values with the opening of the exit cross-section. We discuss to some extent also the value of the heat output necessary to decrease the exit temperature to zero.

2. Problem statement and governing equations

The problem under consideration is depicted in Fig. 1 where the upper half of a symmetric channel, in which the flow is two-dimensional, is sketched by using non-dimensional denotations. It is assumed that the cross-section of the channel varies slowly in x-direction, i.e. that $\alpha_{\max} \equiv \varepsilon \ll 1$ is a small parameter. All physical quantities will vary slowly in x-direction in this case too, and in order to express their slow variations explicitly we will introduce the slow coordinate $\xi = \epsilon x$. Also, the transverse velocity component v will be much smaller than the longitudinal one u everywhere, and we will write: $v(x,y) = \epsilon V(\xi,y)$, where $V(\xi,y)$ is an O(1) transverse velocity. Another small parameter of the same order of magnitude will be $\gamma M_0^2/\mathrm{Re} = \lambda\epsilon, \lambda = \mathrm{O}(1)$, where γ is the ratio of specific heats, and M_0 and Re are the reference Mach number and the reference Reynolds number respectively, defined in the usual way by means of the corresponding physical quantities taken at the origin (see Fig. 1).

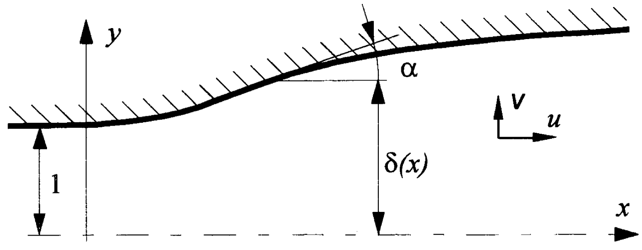


Fig. 1. Upper half of a symmetric 2-D channel.

Continuity equation, ideal gas law, momentum equations in x- and y-directions, and the energy equation, written in non-dimensional form and by using conventional notations read, respectively:

$$\frac{\partial(\rho u)}{\partial \xi} + \frac{\partial(\rho V)}{\partial y} = 0, \quad p = \rho T,$$

$$\lambda\epsilon \,\mathrm{Re}\, \rho\left(u\frac{\partial u}{\partial \xi} + V\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial \xi} + \lambda\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \mathrm{O}(\epsilon^2),$$

$$\frac{\partial p}{\partial y} = \mathrm{O}(\epsilon^2)$$

$$\epsilon \,\mathrm{Re}\,\mathrm{Pr}\, \rho\left(u\frac{\partial T}{\partial \xi} + V\frac{\partial T}{\partial y}\right) = \frac{\gamma-1}{\gamma}\epsilon \,\mathrm{Re}\,\mathrm{Pr}\left(u\frac{\partial p}{\partial \xi} + V\frac{\partial p}{\partial y}\right)$$

$$+\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right) + \frac{\gamma-1}{\gamma}\epsilon\lambda \,\mathrm{Re}\,\mathrm{Pr}\,\mu\left(\frac{\partial u}{\partial y}\right)^2 + \mathrm{O}(\epsilon^2). \tag{1}$$

Here, $\mathrm{Pr} = \mathrm{O}(1)$ is the reference Prandtl number. We assume that a local Prandtl number is constant (independent of the temperature!), so that non-dimensional values of the coefficients of conductivity k and viscosity μ will be equal and will depend on the temperature according to a power law: $k = \mu = T^n, n \geqslant 0$. Our aim is to develop an expansion scheme in which inertia terms in both the momentum equation in the x-direction and the energy equation, as well as the dissipation term in the energy equation can be neglected in the first approximation. For that purpose it is obviously necessary to assume that $\mathrm{Re} = \mathrm{O}(1)$, which can be maintained in microchannels only. If the Reynolds number attains some moderately high values, one should have in mind that: $\lambda\epsilon\,\mathrm{Re} = \gamma M_0^2$, so that the desired form of the first approximation can be achieved for a low Mach number flow in the channel. In both cases M_0 should be small enough for the theory presented here to be valid. Possible solutions of (1) should satisfy the non-slip boundary conditions on the wall,

$$y = \delta(\xi), \quad u = V = 0, \tag{2}$$

and the symmetry conditions on the centreline:

$$y = 0, \quad \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = V = 0. \tag{3}$$

The boundary conditions for the temperature field on the wall will be stated later.

Each physical quantity $f(\xi,y)$ present in (1) will now be expanded into a regular perturbation series of the form:

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$$

In this way, the first order equations read (for simplicity we will drop the subscript "0")

$$\frac{\partial(\rho u)}{\partial \xi} + \frac{\partial(\rho V)}{\partial y} = 0, \quad p = \rho T, \quad \lambda\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\mathrm{d}p}{\mathrm{d}\xi},$$

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial T}{\partial y}\right) = 0, \tag{4}$$

with the boundary and the symmetry conditions:

$$y = \delta(\xi), \quad u = V = 0, \quad y = 0, \quad \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = V = 0. \tag{5}$$

In the first approximation the flow is obviously based on the balance between pressure and viscosity forces as in the classical hydrodynamic lubrication theory, and Eq. (4) represent the so-called "lubrication" approximation. Although non-linear, they are easily amenable to simple analytical methods, as will be shown in the following sections.

3. Isothermal flow

As well known, isothermal gas flows are, strictly speaking, not consistent with the full system of governing equations. However, due to very small temperature variations many gas flows in engineering can be treated as nearly isothermal. Formally, first order equations for an isothermal gas flow in a channel can be obtained from (4) by inserting $T=1$. The momentum equation yields the following solution for the longitudinal velocity:

$$u = -\frac{\delta^2}{2\lambda}\frac{\mathrm{d}p}{\mathrm{d}\xi}\left(1-\frac{y^2}{\delta^2}\right),$$

which is a slowly varying parabola. The continuity equation is a first order equation in V subjected to two conditions (5). In addition to the transverse velocity, it leads to the following equation for the pressure too,

$$p\frac{\mathrm{d}p}{\mathrm{d}\xi}\delta^3 = \mathrm{const}.$$

The physical meaning of this constant can be deduced by the introduction of the mass flow rate through the channel. If $\dot{m}$ is a non-dimensional first order mass flow rate, we will have

$$p\frac{\mathrm{d}p}{\mathrm{d}\xi}\delta^3 = -3\lambda\dot{m}. \tag{6}$$

Thus, the pressure decreases in the direction of flow independently of whether the channel (nozzle!) is convergent or divergent. This is contrary to the well known results of classical Gas Dynamics. However, when judging this pressure drop one has to realise that the flow considered herein is a friction dominated one, and that its behaviour is consequently more reminiscent of a Poiseuille-type flow than that of an inertia dominated classical gas flow through nozzles. Introducing the mass flow rate $\dot{m}$ both velocity components can be written as:

$$u = \frac{3\dot{m}}{2p\delta}\left(1-\frac{y^2}{\delta^2}\right), \quad V = \frac{3\dot{m}}{2p\delta}\frac{\mathrm{d}\delta}{\mathrm{d}\xi}\frac{y}{\delta}\left(1-\frac{y^2}{\delta^2}\right). \tag{7}$$

Integration of Eq. (6) between the inlet station $\xi = 0$ and an arbitrary cross section of the nozzle yields

$$1 - p^2 = 6\lambda\dot{m}\int_0^{\xi}\delta^{-3}(\xi)\,\mathrm{d}\xi, \tag{8}$$

where p is the pressure at the arbitrary cross section. This result can be utilised for a nozzle of given shape and length l to determine the first order exit pressure p_e if mass flow rate is given, or to determine the first order mass flow rate if the exit pressure is given. Clearly, the mass flow rate attains its maximum value for $p_e = 0$, and since the integral (8) does not converge for $\xi \to \infty$, a finite (critical) length of the channel over which the pressure drops to zero will always exist. This phenomenon is called the "mathematical" choking (see Schwartz, 1987). Clearly, convergent nozzles are more prone to choking than divergent ones, in the sense that in convergent nozzles choking occurs over a shorter length. It is to be noted at this point that both velocity components (7) become infinite at the critical cross section of the channel.

We explored this phenomenon in some more details by going to the next approximation. It yields the following differential equation for the pressure p_1:

$$35p^2\delta^3 p_1' - 105\lambda\dot{m}p_1 = -105\lambda\dot{m}_1 p + 54\lambda\,\mathrm{Re}\,\dot{m}^2(p\delta)',$$

(where $\dot{m}$ is the second order mass flow rate), which can be written by using (8) in the following form:

$$35\dot{m}(p\,p_1)' = 35\dot{m}_1 p\,p' + 54\lambda\,\mathrm{Re}\,\dot{m}^3\delta^{-3}\delta' - 162\lambda^2\,\mathrm{Re}\,\dot{m}^4 p^{-2}\delta^{-5}.$$

This can be formally integrated between the inlet ($\xi = 0$) and an arbitrary cross section by assuming that the inlet pressure is known and completely contained into the first approximation $p(\xi)$, so that $p_1(0) = 0$, leading to

$$35\dot{m}p\,p_1 = \frac{35}{2}\dot{m}_1(p^2-1) + 27\lambda\,\mathrm{Re}\,\dot{m}^3\left(1-\frac{1}{\delta^2}\right) - 162\,\lambda^2\,\mathrm{Re}\,\dot{m}^4\int_0^{\xi}p^{-2}\delta^{-5}\,\mathrm{d}\xi. \tag{9}$$

This result, exactly as the one expressed by (8), can be used twofold. If the mass flow rate through the nozzle is given it will be fully embodied into the first order mass flow rate, so that $\dot{m}_1 = 0$, then we can determine the correction of the pressure p_1 from (9). On the contrary, if the exit pressure is given it will be fully incorporated into the first order pressure p_e, so that $p_1(\xi = l) = p_{1e} = 0$, and Eq. (9) serves for the determination of the correction of the mass flow rate $\dot{m}_1$. In both cases the integral on the right-hand side of (9) participates in the solution. It can be readily shown by employing (6) that this integral behaves as $\ln p$ by approaching the critical cross section of the nozzle. Thus, no matter what is given and what is required, p_1 or $\dot{m}_1$ blow up at the critical cross-section, and singularities at this section become even stronger in the second approximation, than in the first one, because they are now associated with the pressure and the mass flow rate too. It is not difficult to reveal the source of singularities in both the first order solution and the second order solution. By approaching the critical cross-section of the channel a local value of (Mach number)^2/Reynolds number, taken say at the centreline, increases indefinitely, which is contrary to one of the basic assumptions of the theory (see Section 1). Thus, in the vicinity of the critical cross-section inertia forces are not negligible, and this vicinity ought to be treated separately, with the equations containing inertia forces in addition to the viscous ones. Possible analytical solutions should then be asymptotically matched with the solutions obtained here, which is beyond the scope of this work. So far, the question concerning the (finite!) value of the velocity at the exit of a channel of critical length is open!

There is another effect, worth to note, which should be taken into consideration in the vicinity of the critical cross-section. This is the rarefaction effect. Namely, due to the significant decrease of pressure that occurs close to the critical cross-section of the channel, a local Knudsen number may attain relatively large value, so that non-slip boundary conditions cannot be applied. The simplest modification of the model considered herein, that can be performed in order to meet the requirements imposed by the rarefied gas dynamics, is the application of slip boundary conditions (for the definition of so-called high order slip boundary conditions see Beskok et al., 1996). Competing effects of compressibility, viscous heating and rarefaction in channels of varying cross section may throw new light on micro-channel gas flow. We hope to be able to publish results on that topic in the near future.

In concluding this Section we will plot the first order mass flow rate through a nozzle consisting of plane walls: $\delta = 1 + (\delta_e - 1)\xi/l$, where δ_e is the exit half width of the nozzle, versus δ_e in Fig. 2. As expected, the mass flow rate increases δ_e, this increase being very pronounced for divergent nozzles ($\delta_e > 1$).

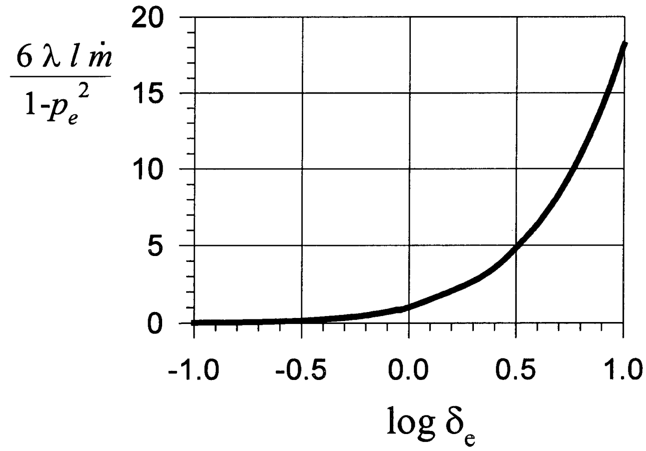


Fig. 2. Mass flow rate versus exit half-width of the channel for an isothermal flow.

4. Non-isothermal flow

Non-isothermal gas flows are more realistic than isothermal ones, and we will discuss here a couple of characteristic problems associated with the application of various boundary conditions on the wall.

The energy equation (4), when the symmetry condition (3) is employed, leads to a simple solution for the first order temperature as $T = T(\xi)$. When we treat the heating/cooling problem in which constant or variable wall temperature $T_w(\xi)$ is given, the solution for the first order temperature will read $T = T_w(\xi)$, which means that the first order heat flux through the wall will be zero, and one has to go to the next approximation in order to evaluate it. When we treat the adiabatic wall problem, the corresponding boundary condition on the wall is automatically satisfied, and $T(\xi)$ represents the so-called eigen temperature, which remains undetermined at this stage. As in the previous case, one has to go to the second order equations to find its value.

The solutions of the rest of Eq. (4) are readily found to be

$$\frac{p\delta^3}{\mu T}\frac{\mathrm{d}p}{\mathrm{d}\xi} = -3\lambda\dot{m}, \quad u = \frac{3\dot{m}}{2}\frac{T}{p\delta}\left(1 - \frac{y^2}{\delta^2}\right),$$

$$V = \frac{3\dot{m}}{2}\frac{T\delta'}{p\delta}\frac{y}{\delta}\left(1 - \frac{y^2}{\delta^2}\right). \tag{10}$$

Also, the second order energy equation can be routinely derived from (1), solved for the second order temperature $T_1(\xi, y)$, and the heat flux through the wall $q(\xi)$ evaluated. In this way, we obtain

$$\mu\frac{\partial T_1}{\partial y}\bigg|_{y=\delta} = \mathrm{Re}\ \mathrm{Pr}\ \dot{m}T' \equiv q. \tag{11}$$

Consequently, in the heating/cooling problem a second order heat flux exists only if the wall temperature varies over the channel. In the adiabatic wall problem $q = 0$, and $T = 1$, which means that this problem and the isothermal flow problem (Section 3) are identical to this order of approximation.

In what follows we will assume that a constant or variable heat flux q is given in advance and we will discuss its influence on the flow characteristics. First, one can be disappointed by the fact that in this problem the heat flux through the wall is a second order effect, although one of major motivations for this and other related works is just the cooling of electronic circuits, superconducting magnets, etc., as stated in the Introduction. However, a relatively small heat flux does not a priori mean that it cannot be usefully exploited, especially if one has in mind the application of helium at 100 K (see Shajii and Freidberg, 1996).

Integration of the equation for the pressure (10) between the inlet station ($\xi = 0$) and an arbitrary cross section of the channel leads to

$$1 - p^2 = 6\lambda\dot{m}\int_0^\xi T^{1+n}\delta^{-3}\,\mathrm{d}\xi, \tag{12}$$

where n is the exponent in the power law dependence of transport coefficients on the temperature. This expression represents an extension of (8) to non-isothermal flows and can be used in exactly the same way. The discussion concerning the phenomenon of "mathematical" choking, performed earlier (see Section 3), applies here too with some modifications. If heat is added to the gas ($q > 0$), the temperature increases along the channel (11) and this makes the critical length of the channel became shorter for a convergent nozzle, and longer for a divergent one. Heat subtraction causes the drop of temperature along the channel and affects the critical length conversely.

We will now focus our attention on the influence of the applied heat flux on the mass flow rate through a channel of given length l and exit pressure $p_e \geqslant 0$. For simplicity we will assume that the heat flux is constant and that the channel consists of plane walls: $\delta = 1 + (\delta_e - 1)\xi/l$. For $q =$ const. the first-order temperature will be a linear function of ξ, as revealed by (11): $T = 1 + Q\xi/\dot{m}$, where $Q = q/\mathrm{Re}\,\mathrm{Pr}$, and the exit temperature is obtained as $T_e = 1 + A$, with $A = Ql/\dot{m} \geqslant -1$. It is readily shown in this case that expression (12), when applied for the exit cross-section of the channel, can be conveniently written in one of the following two forms:

$$\frac{6\lambda l\dot{m}}{1 - p_e^2} = \frac{1}{I(A)} \quad \text{or} \quad \frac{6\lambda l^2 Q}{1 - p_e^2} = \frac{A}{I(A)}, \tag{13}$$

where

$$I(A) = \int_0^1 \frac{(1 + At)^{1+n}}{[1 + (\delta_e - 1)t]^3}\,\mathrm{d}t.$$

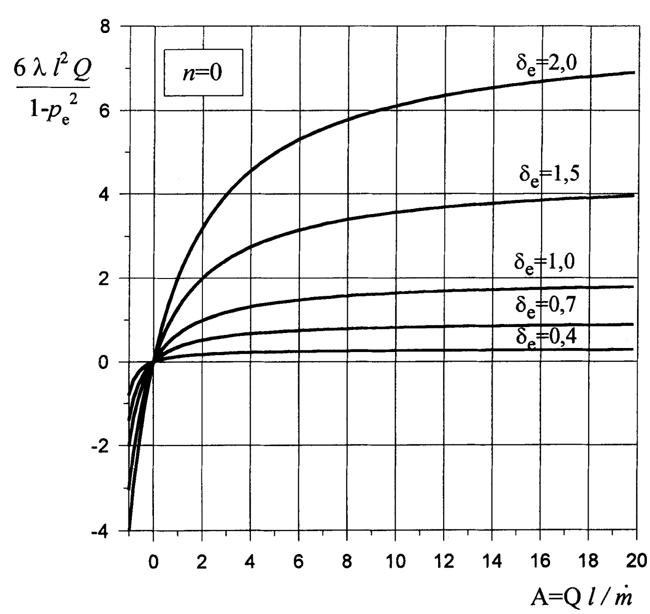
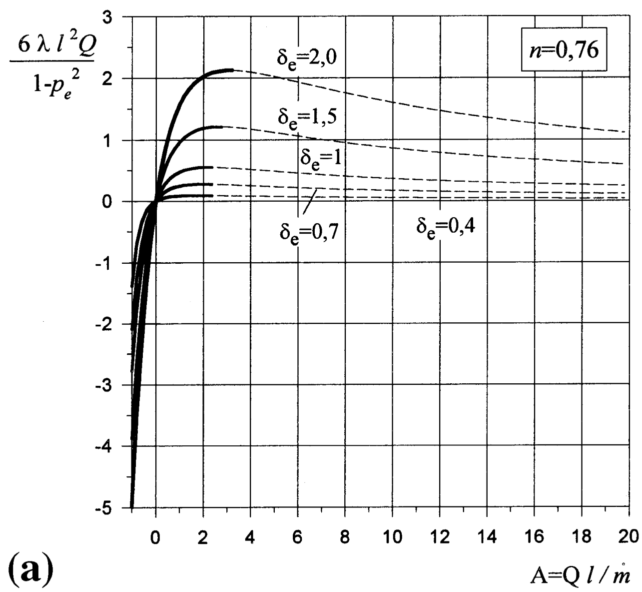


Fig. 3. Dependence of the mass flow rate on the heat flux for $n = 0$.

From the first form we can easily prove that $\partial\dot{m}/\partial A < 0$. The sign of the corresponding derivative $\partial Q/\partial A$ cannot be readily deduced from the second form. However, the only physically acceptable flow regime is one in which the exit temperature increases by adding the heat to the flow, which means that $\partial Q/\partial A > 0$. Consequently, we will have $\partial\dot{m}/\partial Q < 0$, and the mass flow rate will decrease with the heat input. This is qualitatively contrary to the well known result of classical (inviscid!) Gas Dynamics (Rayleigh flow).

When $6\lambda l^2 Q/(1-p_e^2)$ is plotted versus $A = Ql/\dot{m}$, the results obtained by numerical evaluation of $I(A)$ show some qualitative dependence on n. For $n=0$ and different values of δ_e they are presented in Fig. 3.

All curves tend to some constant values as $A \to \infty$ – the mass flow rate drops to zero at some finite critical heat input



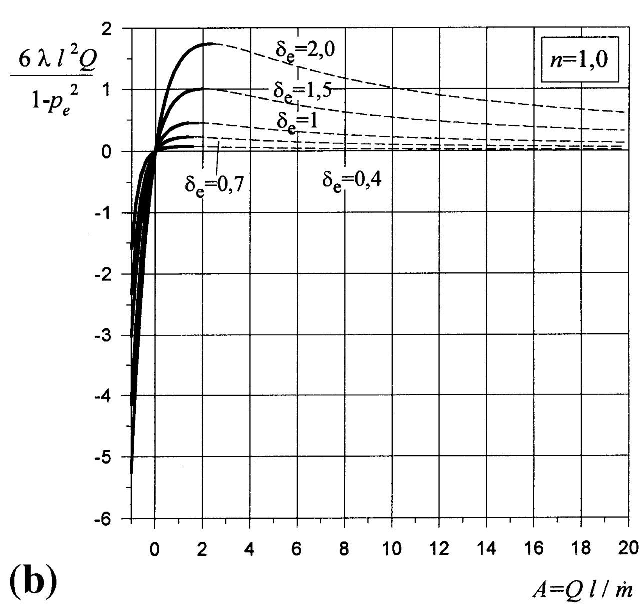


Fig. 4. (a) Dependence of the mass flow rate on the heat flux for $n=0.76$ (b) Dependence of the mass flow rate on the heat flux for $n=1$.

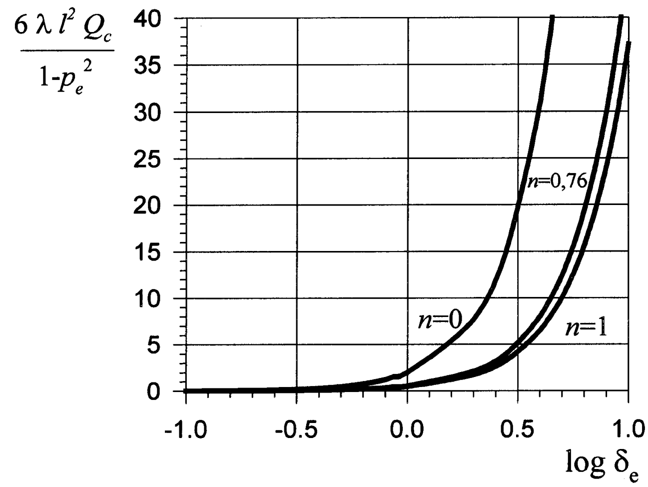


Fig. 5. Critical heat input versus exit half-width of the channel.

Q_c, which increases with the opening of the channel δ_e. Also, some finite heat subtraction from the flow, which increases with δ_e too, is necessary to reduce the exit temperature to zero ($A=-1$). For $n > 0$ the situation is physically different in that A as a function of Q is a double-valued function (see Fig. 4(a) and (b)). The right (dotted!) branch should be rejected, however, because the exit temperature would decrease along this branch with the addition of heat.

In addition to the critical heat flux there is now a non-zero critical mass flow rate $\dot{m}_c$. Critical heat flux is much smaller for any δ_e than in the case $n=0$, while the corresponding value of the heat output necessary to reduce the exit temperature to zero is greater. In Figs. 5 and 6 we plot critical values of heat input and mass flow rate, $6\lambda l^2 Q_c/(1-p_e^2)$ and $6\lambda l^2\dot{m}_c/(1-p_e^2)$, versus $\log\delta_e$ for different n, respectively. They both increase with δ_e, the increase being particularly pronounced for divergent nozzles. Since critical mass flow rate is zero for $n=0$, it naturally increases with n. The opposite trend holds for the critical heat flux.

A natural question which can be put forward at this point is related to the flow regime for which $Q > Q_c$. This regime was discussed in details in Shajii and Freidberg (1996), and it was

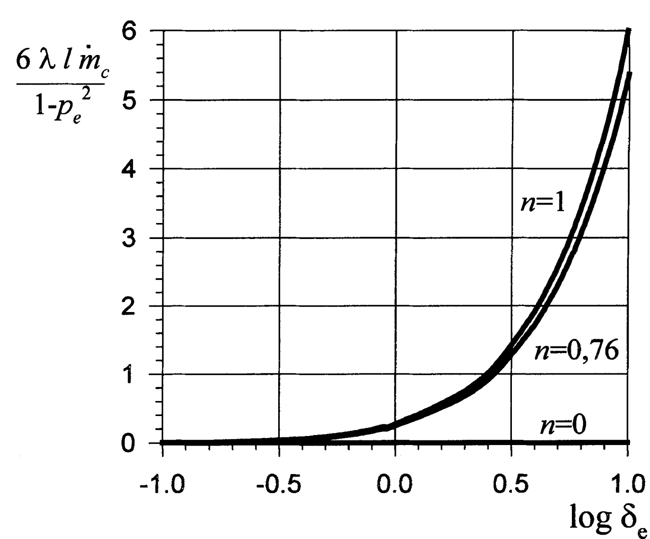


Fig. 6. Critical mass flow rate versus exit half-width of the channel.

found that a steady flow cannot be maintained in this case. Numerical integration of unsteady flow equations shows that the gas is perpetually depleted from both the inlet and the outlet cross-section of the pipe. We do not have any reason to doubt that the situation in our case concerning this point should be different, and since our primary aim in the paper is the study of the effect of variable cross-section of the channel on the characteristics of low Mach number flows, we will omit the corresponding analysis of the regime $Q > Q_c$ from the contents.

5. Conclusions

The analytical treatment of low Mach number, viscous compressible flow in micro-channels of slowly varying cross-section, in this paper represents a rational extension of the previously performed analyses of isothermal flow between parallel plates (see Schwartz, 1987), and non-isothermal flow with given heat flux in a pipe (see Shajii and Freidberg, 1996). All qualitatively important results obtained earlier have been confirmed on the example of the flow treated in this paper, and the effect of varying cross-section of the channel upon flow characteristics discussed. Therefore, the following major conclusions can be withdrawn:

(a) The phenomenon of "mathematical" choking, consisting in the pressure drop to zero and in the indefinite increase of velocity components at some finite critical length of the channel, is influenced by the variable cross-section in isothermal flow, in such a way that the critical length is much longer in divergent nozzles, than in convergent ones. In non-isothermal flow with heat addition this effect is even more pronounced.

(b) Isothermal and adiabatic wall problems are identical form the point of view of the first order equations.

(c) Heat addition in non-isothermal flow case causes the decrease of the mass flow rate through the channel, which is contrary to the Rayleigh flow known from the classical inviscid Gas Dynamics. The steady regime of flow is possible only if the heat flux is less than a critical value. If transport coefficients are temperature independent, the critical heat flux reduces the mass flow rate to zero, and stops the flow in the channel. If their dependence on the temperature is pronounced, the critical heat flux reduces the mass flow rate to a non-zero positive value – the critical mass flow rate. Both critical heat input and critical mass flow rate increase with the width of the channel exit section, and this increase is very pronounced in divergent channels.

(d) A finite heat subtraction is necessary to reduce the exit temperature to zero. It increases with the opening of the channel.

Engineers are usually prone to treat the low Mach number compressible viscous flow as the incompressible one, by using some appropriately defined values of the friction coefficient and the heat transfer coefficient for a laminar incompressible flow. The results presented here and by previous authors, show that such an analogy is not consistent with the theory. Both flows differ quantitatively as well as qualitatively, so that great mistakes can be made if the theory of incompressible flows is applied for calculation of low Mach number compressible flows.

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